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On long-wave sound scattering by a Rankine vortex: Non-resonant and resonant cases

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ABSTRACT

The well-known two-dimensional problem of sound scattering by a Rankine vortex at small Mach number M is considered. Despite its long history, the solutions obtained by many authors still are not free from serious objections. The common approach to the problem consists in the transformation of governing equations to the d'Alembert equation with right-hand part. It was recently shown [I.V. Belyaev, V.F. Kopiev, On the problem formulation of sound scattering by cylindrical vortex, Acoustical Physics 54(5) (2008) 603–614] that due to the slow decay of the mean velocity field at infinity the convective equation with nonuniform coefficients instead of the d'Alembert equation should be considered, and the incident wave should be excited by a point source placed at a large but finite distance from the vortex instead of specifying an incident plane wave (which is not a solution of the governing equations).

Here we use the new formulation of Belyaev and Kopiev to obtain the correct solution for the problem of non-resonant sound scattering, to second order in Mach number M. The partial harmonic expansion approach and the method of matched asymptotic expansions are employed. The scattered field in the region far outside the vortex is determined as the solution of the convective wave equation, and van Dyke's matching principle is used to match the fields inside and outside the vortical region. Finally, resonant scattering is also considered; an $O(M^2)$ result is found that unifies earlier solutions in the literature. These problems are considered for the first time.

1. Introduction

The problem of sound scattering by a two-dimensional circular vortex has attracted considerable attention [1–14], since it is a basic problem in the theory of sound interaction with low Mach number flows. The case in point is the scattering of a plane sound wave by a cylindrical Rankine vortex. The literature contains several different results for this problem each claiming to be correct; meanwhile, an analysis of what is wrong with the others is not usually provided.

The question of a consistent formulation for the problem of sound scattering by the vortex is considered in detail in [1], where it is shown that higher-order terms in M must be retained in the wave operator in order to provide an unambiguous description of the incident field. Thus within the framework of the plane wave scattering formulation (which reflects the use of the d'Alembert wave operator), it is difficult to capture the physical essence of the problem.

Three main approaches can be distinguished. The first is based on the solution of Lighthill's equation [2–4,6,8,11,13]; the second makes use of Howe's equation [5,14], for the stagnation enthalpy; and the third approach is based on a partial

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harmonic expansion of the incident sound field [7,9,10,12]. In these papers a plane wave is used as the incident field, which leads to implicit or explicit difficulties. This is due to the fact that the plane wave is not a solution of the governing equations at any distance from the vortex, even if the Mach number is small, because the mean velocity field decays at infinity too slowly; this question is considered in detail in [1]. Transformation of the governing equations into the form where the left-hand side is d'Alembert's operator and all the difficulties are translated to the right-hand side (the source), which is the first approach and the one used in most papers, is not suitable for sound scattering by a vortex, owing to the slow decay of the source at infinity. The solution involves integrals that do not exist in the general sense, and thus depend upon the method of integration. For example, the solutions obtained by using a polar coordinate system for integral evaluation [2–4,6,8] and a Cartesian coordinate system [11,13] are different.

It may seem that the problem could be solved by using Howe's equation. Indeed, if one confines attention to the component of the scattered field emitted by the oscillation of the vortex under plane-wave forcing (see [5]), one will obtain well-defined integrals and thus a mathematically good solution; however, in this case the part of the scattered field due to the transformation of the incident wave by the slowly decaying mean velocity field of the vortex is neglected. An attempt to account for this part by reducing the Blokhintsev–Howe operator to d'Alembert's operator and translating the refracting term from the left of the equation to the right [14] nullifies the advantage of Howe's equation, because the translated term (the source) possesses the same drawbacks as in the first approach (see discussion in [1] of Sakov's approach [11]), so that the solution is determined by integrals that do not exist in the general sense and thus depend upon the method of integration. Difficulties of the same nature arise when the third approach is employed and a plane wave is used as the incident field. Note that the ill-posedness of the incident plane wave problem is not universally accepted.

Therefore, in [1] it is proposed to consider as the incident field the field of a point harmonic source placed at large but finite distance R from the vortex in the mean flow around the vortex (Fig. 1). A solution of the governing equations in the leading and the second approximation in M for the point source is to be chosen as an incident field. In the new formulation, the correct asymptotic solution to the problem of non-resonant sound scattering by a Rankine vortex in a weakly compressible fluid is obtained. The specific region I: $\alpha/M \ll r \ll R$ is considered where the comparison of this result with existing solutions is provided. It turns out that the majority of papers [2–6,10–14] correctly describe the sound–vortex interaction field (excluding a narrow parabolic region behind the vortex) by representing it as the sum of a plane wave and a correction term, proportional to

$$f(\theta) \sim M^2 \left(\sin\theta - \cot\frac{\theta}{2} \right) \quad (1)$$

The term $\cot(\theta/2)$, which is singular in the direction $\theta=0$ behind the vortex, describes the distortion of the incident field in the slowly decaying mean velocity field outside the vortex core. The proper solution is non-singular, and the solution (1) breaks down in a narrow parabolic region behind the vortex. The correct wavefront deformation is obtained in [11,13–14]. However, all these comparisons can be made only in region I, whereas outside it all of the existing solutions are invalid.

It should be noted that a point source model is also used in [4]. However, that paper focuses mostly on the case of a point source *near* the vortex. The case of the point source far from the vortex is, by analogy with usual diffraction problems, substituted in [4] by the problem of plane wave scattering, i.e. the case of the point source at large but finite distance from the vortex has not, in fact, been investigated.

The present paper also addresses the important question of resonant scattering by the vortex. In Refs. [9,10] it is demonstrated that when the incident sound frequency coincides with a resonance frequency (i.e. the real part of an

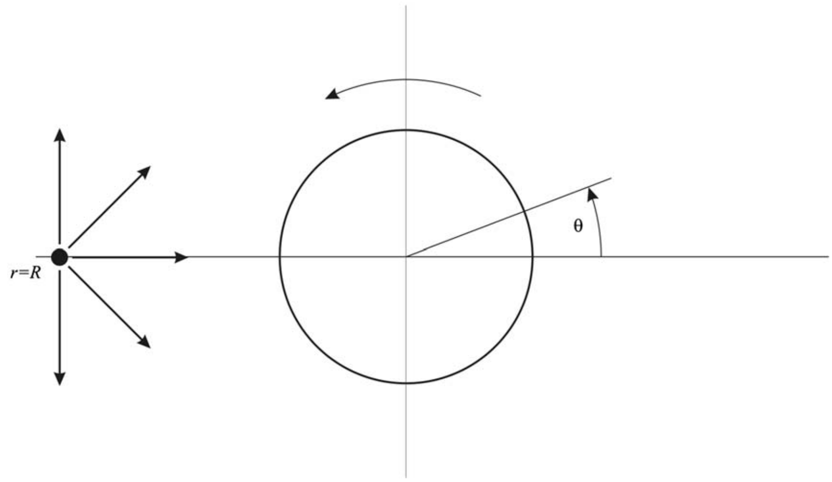

Fig. 1. The Rankine vortex—a two-dimensional vortex of constant vorticity in the circle of radius $r=\alpha$. Point harmonic source placed at large but finite distance R from the vortex.

eigenfrequency) of the vortex, the scattered pressure amplitude becomes $O(1)$, and not $O(M^2)$ as in the case of non-resonant scattering. This result relates to the presence of poles in the scattering amplitude for the eigenfrequencies of the system, and is a general result of the scattering theory. However, it has been argued in [15], by using matched asymptotic expansions up to $O(M^2)$ terms, that although the scattered amplitude increases at the resonance frequency, it remains small and does not exceed $O(M^2)$. It turns out that both statements [10,15] are correct. The resonant scattering amplitude may indeed be of order unity, as is obtained in [9,10]. However, this takes place not at the incompressible vortex eigenfrequency, but at the exact compressible vortex frequency, which differs from the incompressible vortex frequency by terms of $O(M^2)$. If the incompressible vortex eigenfrequency is considered, one obtains the result found in [15].

The structure of the paper is as follows. Section 2 defines the problem and sets out the governing equations. The solution for non-resonant scattering of sound due to a point source at large but finite distance is obtained in Section 3. The results for resonant sound scattering are discussed in Section 4, where an expression for the scattering amplitude is provided that unifies the results of previous researchers and resolves the contradiction between them.

2. The governing equations

Let there be a Rankine vortex of radius α in an ideal (inviscid and non-heat-conducting) compressible fluid. In the cylindrical coordinate system (r,θ,z), this means that inside the radius $r=\alpha$, the z-component of vorticity is a constant Ω_0 and the other components are zero. There is no vorticity outside $r=\alpha$. The flow is supposed to be homentropic and independent of z. Expressions for the mean velocity are well known and coincide with those for the incompressible flow problem. Namely, the radial vorticity component is zero and the azimuthal component V_0 is

$$V_0=\frac{\Omega_0 r}{2},\quad r<\alpha \text{ and } V_0=\frac{\Omega_0\alpha^2}{2r},\quad r\geq\alpha. \tag{2}$$

By assuming the fluid is an ideal gas, with constant specific-heat ratio γ, explicit expressions can be found for the unperturbed pressure P_0 and density τ_0. The dependence of P_0 and τ_0 on the coordinate r differs from that of the incompressible flow problem, so that the squared speed of sound $a_0^2(r)$ is as follows:

In the region $r\geq\alpha$

$$a_0^2(r)=\gamma\frac{P_0(r)}{\tau_0(r)}=a_0^2(\alpha)\left[1+\frac{\gamma-1}{2}M^2\left(1-\frac{\alpha^2}{r^2}\right)\right] \tag{3}$$

In the region $r<\alpha$

$$a_0^2(r)=\gamma\frac{P_0(r)}{\tau_0(r)}=a_0^2(\alpha)+\frac{\gamma-1}{2}\Omega_0^2(r^2-\alpha^2) \tag{4}$$

Here, $a_0(\alpha)$ is the sound speed at the core boundary $r=\alpha$, and $M=\Omega_0\alpha/2a_0(\alpha)$ is the Mach number at this radius.

Since it is an acoustic scattering problem we are interested in, it is necessary to find the governing equations for the propagation of small pressure perturbations of the form $\exp(-\mathrm{i}\omega_0 t)$ through the regions inside and outside the vortex, ω_0 being the angular frequency of the incident sound wave. The problem will be solved for the pressure perturbation in the linear approximation. The sound wavelength λ is assumed to be long ($\lambda\gg\alpha$) and the Mach number is assumed to be small ($M\ll 1$). The solution will be sought as an expansion in powers of M.

We decompose the velocity, pressure and density fields as follows: $\mathbf{V}_0+\hat{\mathbf{u}}$, $P_0+\hat{p}$, $\tau_0+\hat{\rho}$, where $\hat{\mathbf{u}}$, $\hat{p}$ and $\hat{\rho}$ are the velocity, pressure and density perturbation fields, respectively. Substituting them into the Euler equations and taking (2)–(4) into account, we get the following system of equations in the linear approximation:

$$\begin{cases}\dfrac{\partial\hat{u}_r}{\partial t}+\dfrac{V_0}{r}\dfrac{\partial\hat{u}_r}{\partial\theta}-\dfrac{2V_0}{r}\hat{u}_\theta=\dfrac{\hat{\rho}}{\tau_0^2}\dfrac{\partial P_0}{\partial r}-\dfrac{1}{\tau_0}\dfrac{\partial\hat{p}}{\partial r}\\ \dfrac{\partial\hat{u}_\theta}{\partial t}+\dfrac{V_0}{r}\dfrac{\partial\hat{u}_\theta}{\partial\theta}+\left(\dfrac{\partial V_0}{\partial r}+\dfrac{V_0}{r}\right)\hat{u}_r=-\dfrac{1}{\tau_0 r}\dfrac{\partial\hat{p}}{\partial\theta}\\ \dfrac{\partial\hat{\rho}}{\partial t}+\dfrac{1}{r}\dfrac{\partial}{\partial r}(\tau_0 r\hat{u}_r)+\dfrac{\tau_0}{r}\dfrac{\partial\hat{u}_\theta}{\partial\theta}+\dfrac{V_0}{r}\dfrac{\partial\hat{\rho}}{\partial\theta}=0\\ \hat{p}=a_0^2\hat{\rho},\quad a_0^2=\gamma P_0/\tau_0\end{cases} \tag{5}$$

Here, $\hat{u}_r$ and $\hat{u}_\theta$ are the radial and azimuthal components of the velocity $\hat{\mathbf{u}}$, and $a_0^2(r)$ is given by (3) or (4). The solution for the linearized Euler equations will be sought as a sum of azimuthal harmonics: thus $\hat{p}=\mathrm{e}^{-\mathrm{i}\omega_0 t}\sum_{n=-\infty}^{+\infty}\hat{p}_n(r)\mathrm{e}^{\mathrm{i}n\theta}$ for the pressure perturbation, and similarly for the density and the components of velocity. Eventually the real part of the solution is required; but because the whole analysis is linear, taking the real part can be deferred until it is necessary to discuss energy fluxes.

It is convenient to introduce dimensionless variables x, ω, $a^2(x)$ such that $r=\alpha x$, $\omega_0=\omega\Omega_0/2$, $a_0^2(r)=a_0^2(\alpha)a^2(x)$; the azimuthal harmonic amplitudes of pressure, velocity and density are written as $\hat{p}_n=p_nP_0(\alpha)$, $\hat{\mathbf{u}}=(\Omega_0\alpha/2)\mathbf{u}$, $\hat{\rho}_n=\rho_n\tau_0(\alpha)$, respectively. The solution of the linearized problem for each azimuthal harmonic n will be found as follows. We distinguish two main regions: (i) inside the vortex, $x\sim 1$ and (ii) outside the vortex, $x\geq 1$; the latter is partitioned into a near-field subregion, $x\sim 1$ and a far-field subregion $x\gg 1$, which overlap in an intermediate subregion. The condition for the incident field (e.g. a plane wave or a point source) is posed in the far-field subregion. The solution in the far-field subregion (a sum of the incident field and the scattering field) is to be matched with the solution in the near-field subregion. The near-field subregion solution (the pressure and normal velocity) must coincide at the boundary of the vortex with the solution inside the vortex, which is finite at the point of origin. These conditions taken together give the complete solution of the problem for each nth harmonic. For the non-resonant case only a two-term expansion in M is required: namely, the terms of $O(1)$ and $O(M^2)$. The solution is a sum of these terms over all harmonics. The resonant case requires more terms in the expansion but for only one azimuthal harmonic, which is resonant.

2.1. Region inside the vortex core

In the core region $r \le \alpha$, the mean velocity and the sound speed are described by (2) and (4), respectively. Taking this into account, from system (5) we obtain for the azimuthal harmonic amplitudes

$$\begin{cases} \dfrac{d^2\hat{p}_n}{dr^2}+\dfrac{1}{r}\left(1-\dfrac{\Omega_0^2r^2}{4a_0^2}\right)\dfrac{d\hat{p}_n}{dr}+\left(\dfrac{\Omega_0^2r^2}{4a_0^4}\dfrac{da_0^2}{dr}-\dfrac{3\Omega_0^2}{2a_0^2}+\dfrac{n\Omega_0^3}{4\sigma a_0^2}+\dfrac{\sigma^2}{a_0^2}-\dfrac{n^2}{r^2}\right)\hat{p}_n=0 \\ \hat{u}_r^{(n)}=\left(\dfrac{\sigma}{\sigma^2-\Omega_0^2}\right)\left[\dfrac{i}{\tau_0}\dfrac{d\hat{p}_n}{dr}+\dfrac{i\hat{p}_n}{\tau_0}\left(\dfrac{n\Omega_0}{r\sigma}-\dfrac{\Omega_0^2 r}{4a_0^2}\right)\right] \\ \hat{u}_\theta^{(n)}=\dfrac{n}{\sigma r}\dfrac{\hat{p}_n}{\tau_0}-\dfrac{i\Omega_0}{\sigma}\hat{u}_r^{(n)} \end{cases} \tag{6}$$

where $\sigma=(n\Omega_0/2)-\omega_0$, and $a_0^2=a_0^2(r)$ is given by (4). This system is well known and can be found, for instance, in [16]. It is supposed here that $\sigma\neq 0$ and $\sigma^2\neq\Omega_0^2$.

The exact equation for the dimensionless amplitude of the nth harmonic p_n is as follows:

$$\frac{d^2p_n}{dx^2}+\frac{1}{x}\left(1-\frac{M^2x^2}{a^2(x)}\right)\frac{dp_n}{dx}+\left[\frac{M^2\Psi}{a^2(x)}+(\gamma-1)\frac{M^4x^2}{a^2(x)}-\frac{n^2}{x^2}\right]p_n=0, \tag{7}$$

where

$$\Psi=(n-\omega)^2-6+\frac{2n}{n-\omega}\equiv const,$$

and

$$a^2(x)=1+\frac{\gamma-1}{2}M^2(x^2-1).$$

After neglecting the $O(M^2)$ terms, one gets

$$\frac{d^2p_n}{dx^2}+\frac{1}{x}(1-M^2x^2)\frac{dp_n}{dx}+\left[M^2\Psi-\frac{n^2}{x^2}\right]p_n=0 \tag{8}$$

Once the solution for Eq. (18) is found, the nth harmonic velocity components $u_r^{(n)}$ and $u_\theta^{(n)}$ are obtained straightforwardly from (6). As noted above, in the case of non-resonant scattering we have to retain terms up to $O(M^2)$; these are determined by Eq. (8). The resonant case requires retaining more terms from the exact equation (7).

2.2. Region outside the vortex core

In the outer region $r\ge\alpha$, sound propagates in an irrotational flow. Therefore we can introduce the velocity potential $\hat{\varphi}:\hat{u}_r=\partial\hat{\varphi}/\partial r,\ \hat{u}_\theta=(1/r)\partial\hat{\varphi}/\partial\theta$. The potential is also sought as a sum of azimuthal modes: $\hat{\varphi}=e^{-i\omega_0 t}\sum_{n=-\infty}^{+\infty}\hat{\varphi}_n(r)\,e^{in\theta}$.

Substituting these formulae into (5), we obtain for the harmonic amplitudes

$$\begin{cases} \dfrac{d^2\hat{\varphi}_n}{dr^2}+\dfrac{1}{r}\left(1+\dfrac{V_0^2}{a_0^2}\right)\dfrac{d\hat{\varphi}_n}{dr}+\left(\dfrac{\sigma^2}{a_0^2}-\dfrac{n^2}{r^2}\right)\hat{\varphi}_n=0 \\ \hat{\rho}_n=i\sigma\dfrac{\tau_0}{a_0^2}\hat{\varphi}_n,\ \hat{p}_n=i\tau_0\sigma\hat{\varphi}_n,\ \sigma(r)\equiv\omega_0-\dfrac{\Omega_0\alpha^2}{2r^2}n \end{cases} \tag{9}$$

Let the potential $\hat{\varphi}_n$ be scaled by $V_0(\alpha)\alpha\equiv\Omega_0\alpha^2/2\equiv M\alpha a_0(\alpha)$. Thus, the exact equation for the dimensionless nth harmonic amplitude φ_n is as follows:

$$\frac{d^2\varphi_n}{dx^2}+\frac{1}{x}\left(1+\frac{M^2}{x^2a^2(x)}\right)\frac{d\varphi_n}{dx}+\left(\frac{M^2\sigma^2(x)}{a^2(x)}-\frac{n^2}{x^2}\right)\varphi_n=0, \tag{10}$$

where

$$\sigma(x)=\omega-\frac{n}{x^2},\quad \alpha^2(x)=1+\frac{\gamma-1}{2}M^2\left(1-\frac{1}{x^2}\right).$$

This equation can be found in [17].

Outside the vortex we distinguish two asymptotic regions: the near-field region, $x \sim 1$ and the far-field region $x > 1/M \gg 1$. These overlap in the intermediate region. The condition for the incident field (e.g. a plane wave) is posed in the far-field subregion.

2.2.1. Near-field region $x \sim 1$

Here we seek the solution of Eq. (10) in the region near the vortex ($x \sim 1$). Neglecting terms of $O(M^4)$ and higher in Eq. (10), one obtains

$$\frac{\partial^2 \varphi_n}{\partial x^2} + \frac{1}{x}\left(1 + \frac{M^2}{x^2}\right)\frac{\partial \varphi_n}{\partial x} + \left(M^2\omega^2 + M^2\frac{n^2}{x^4} - \frac{\nu_0^2}{x^2}\right)\varphi_n = 0, \quad \nu_0^2 = n^2 + 2nM^2 \tag{11}$$

This equation takes into account all $O(M^2)$ terms and holds for distances up to $x \sim 1/M$.

2.2.2. Far-field subregion, $x > 1/M \gg 1$

Neglecting terms of $O(M^4)$ and higher in (10) in the far-field region for $x > 1/M$ we obtain

$$\frac{d^2\varphi_n}{dx^2} + \frac{1}{x}\frac{d\varphi_n}{dx} + \left(k^2\alpha^2 - \frac{|\nu|^2}{x^2}\right)\varphi_n = 0, \quad \text{where } k^2 = \frac{M^2\omega^2}{\alpha^2\left(1 + \frac{\gamma-1}{2}M^2\right)} \tag{12}$$

$$|\nu| = |n| + \beta\,\mathrm{sgn}(n), \quad \beta = \omega M^2 \text{ for } n \neq 0, \quad \nu = 0, \quad n = 0 \tag{13}$$

In Eq. (12) the exact expression for k^2 is preserved. For matching we introduce the outer variable $\xi = k\alpha x \sim Mx$. Two terms of the outer solution expansion can be found from the equation

$$\frac{d^2\varphi_n}{d\xi^2} + \frac{1}{\xi}\frac{d\varphi_n}{d\xi} + \left(1 - \frac{|\nu|^2}{\xi^2}\right)\varphi_n = 0. \tag{14}$$

Eqs. (8), (11) and (12) or (14) are basic for the problem of sound scattering by the Rankine vortex in the non-resonant case and for our analysis of different formulations of the problem. They allow one to seek a matched asymptotic solution, taking the terms of $O(1)$ and $O(M^2)$ into account in the inner region $x \le 1/M$ and in the outer region $\xi \ge 1$, which overlap in the region $x \sim 1/M$. Note that the $O(M^4)$ term $-\omega^2 M^4 \varphi_n / r^2$ has been added in Eq. (12) to replace the exact expression $\nu_0^2 = n^2 + 2nM^2$ in (11) by $\nu^2 = n^2 + 2n\omega M^2 + \omega^2 M^4$. This does not affect the accuracy of our analysis and corresponds to renormalizing the main variable ξ [20]. Eq. (12) will be used in Section 3 for the far-field solution. For van Dyke's matching procedure the same equation, rewritten in outer variables as (14), will be used. Note that Eqs. (12) and (13) appear to be identical to the equation considered by Berry et al. [21]. The simplification above thus allows us to use the approach of [21] for summing the series, with some modification to deal with the more complex series of the present formulation. The ambiguity associated with earlier formulations can thereby be resolved.

In order to consider resonant scattering, higher order terms should be retained, although only for the harmonic n which is excited in the case of resonance. For this n the exact Eq. (7) inside the vortex and the exact Eq. (10) in the near-field subregion outside the vortex will be considered. The solution obtained by joining these solutions is then asymptotically matched with the solution of the far field.

3. Non-resonant scattering

3.1. Incident field

Let there be a point source of sound (a mass source) at the point $r = R, \theta = \pi$. Thus on the right of the third equation in (5) (the continuity equation) there is, instead of zero, the function $q e^{-i\omega_0 t}\delta(r-R)\delta(\theta-\pi)/R$, where q is the density source strength, which from now on is set to be $\Omega_0\tau_0/2$. Let the sound source be located in the region far from the vortex ($R/\alpha \gg 1$); therefore, the harmonic amplitudes for the potential are described by Eq. (12) or Eq. (14). Since $\delta(\theta-\pi) = (1/2\pi)\sum_{n=-\infty}^{+\infty} e^{in(\theta-\pi)}$, the equation for the amplitude φ_n is as follows:

$$\frac{d^2\varphi_n}{dx^2} + \frac{1}{x}\frac{d\varphi_n}{dx} + \left(k^2\alpha^2 - \frac{|\nu|^2}{x^2}\right)\varphi_n = \frac{\alpha e^{-i\pi n}}{2\pi R}\delta(x - R/\alpha). \tag{15}$$

Its solution is $\varphi_n = (\alpha e^{-i\pi n}/2\pi R) G_n(r,R)$, where $G_n(r,R)$ is the Green's function of Eq. (14), given by

$$G_n(x,R) = \frac{\pi R}{2i\alpha}\begin{cases} H_{|\nu|}^{(1)}(kR) J_{|\nu|}(k\alpha x), & 1 \le x \le R/\alpha \\ J_{|\nu|}(kR) H_{|\nu|}^{(1)}(k\alpha x), & R/\alpha \le x \end{cases}$$

It is clear that for $r \ge R$ the solution is an outgoing wave, i.e. it satisfies the Helmholtz condition, which states that as $r \to \infty$ only outgoing waves are present. In the region $\alpha \ll r < R$, the sound field Φ generated by the source is

$$\Phi = \sum_{n=-\infty}^{+\infty} \frac{e^{-i\pi n}}{4i} H_{|\nu|}^{(1)}(kR) J_{|\nu|}(kr)\, e^{in\theta - i\omega_0 t}. \tag{16}$$

This solution (16) will be considered as the incident sound field. Generally, to find the sum of series (16) is a formidable task even if β is supposed small. However, it turns out that it is possible to determine the sum of series (16) in the region $\alpha/M < r \ll R$ (shown blue in Fig. 3; for brevity, this part of the far-field region will be called region I). The method is a modification of that described in [21], which was suggested in [1]. First of all, note that the remainder of the series for $|n| > kr$ can actually be neglected, because kr is large in region I and the Bessel functions $J_{|\nu|}(kr)$ for these n are exponentially small compared with their values for $|n| < kr$. The series (16) is dominated by a finite number of terms, namely those for which $|n| < kr$. For this range of n, since kR is large, we can moreover substitute the Hankel functions by their large-argument asymptotic form,

$$H_\nu^{(1)}(kR) \sim \sqrt{\frac{2}{\pi kR}} e^{i(kR-(\pi\nu/2)-(\pi/4))}.$$

Note that the terms in (16) for which $|n| > kr$ remain exponentially small after the substitution; so approximating the Hankel functions by their far-field asymptotic form involves an exponentially small error. The sound field generated by the source is therefore estimated as

$$\Phi = \sum_{n=-\infty}^{+\infty} \frac{e^{-i\pi n}}{4i} J_{|\nu|}(kr) \sqrt{\frac{2}{\pi kR}} e^{i(kR-(\pi|\nu|/2)-(\pi/4))}\, e^{in\theta - i\omega_0 t} \tag{17}$$

We use the integral representation of the Bessel function (Schläffli integral, see [22])

$$J_\nu(kr) = \frac{1}{2\pi} \int_C \exp\{i(\nu\xi - kr\sin\xi)\}\, d\xi \tag{18}$$

where the integration is taken over the contour C (see the green contour in Fig. 2), which lies above the real axis. Thus, $\operatorname{Im}\xi > 0$ everywhere on the contour.

Substituting the Bessel functions in series (17) by the integral representation (18), we obtain

$$\iota = \frac{e^{-i\omega_0 t}}{8\pi i} \sqrt{\frac{2}{\pi kR}} e^{i(kR-(\pi/4))} \int_C e^{-ikR\sin\xi} \left(\sum_{n=-\infty}^{\infty} e^{i|\nu|\xi + in(\theta-\pi) - i(\pi|\nu|)/2} \right) d\xi$$

The series in the integrand converges uniformly, because $\operatorname{Im}\xi > 0$ for the entire contour; thus, the change in the order of the summation and integration is valid. After evaluating the series for even and odd n separately, we obtain

$$\Phi = \frac{K}{2\pi} \int_C e^{-ikr\sin\xi} f(\xi,\theta)\, d\xi \tag{19}$$

Here, the following notations are introduced:

$$f(\xi,\theta) = 1 + e^{i|\beta|(\xi-(\pi/2))} \frac{e^{i(\theta+\xi+(\pi/2))}}{1-e^{i(\theta+\xi+(\pi/2))}} + e^{-i|\beta|(\xi-(\pi/2))} \frac{e^{i(\xi-\theta+(\pi/2))}}{1-e^{i(\xi-\theta+(\pi/2))}},$$

$$K = \frac{e^{-i\omega_0 t}}{4i} \sqrt{\frac{2}{\pi kR}} e^{i(kR-(\pi/4))} \tag{20}$$

Since kr is large in region I, the saddle-point method can be used to calculate integral (19), according to which the integral is dominated by its saddle points and the poles that are crossed by the contour during its deformation from the

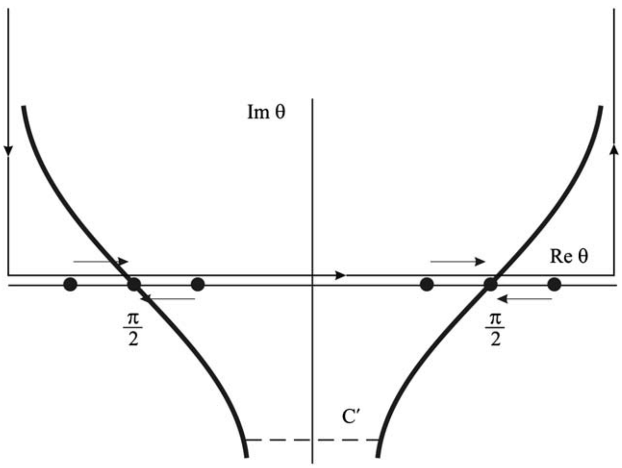

Fig. 2. Initial contour (green), SDP contour (black) and the poles of the integrand.

initial one to the steepest descent path (SDP). The saddle points are $\xi_s = \pm\pi/2$; the parts of the steepest descent path passing through the saddle points are depicted in Fig. 2 by a solid bold black line. Using these parts of the SDP, we can construct the new contour C', so that the integration over the new contour does not change the value of (19). It consists of the SDP part passing through the saddle point $\xi_s = -\pi/2$ from above and the SDP part passing through the saddle point $\xi_s = \pi/2$ from below (this direction corresponds to the direction of the increase of $\mathrm{Re}\,\xi$). For the new contour to be continuous, both parts of the SDP are to be joined, as depicted in Fig. 2 by the dashed line, in the region where the value of the integral is exponentially small. The resulting contour C' consists of the two parts of the SDP and the dashed line (see Fig. 2). The integral over the dashed line and the parts of the SDP below the dashed line can be made arbitrary small if the connection is made far enough from the real axis. Since during the deformation from the initial contour C to the final contour C' it crosses the poles of integrand $f(\xi,\theta)$, integral (19) can be expressed as follows:

$$\Phi = I_{\mathrm{SDP}} - 2\pi\mathrm{i}\sum \mathrm{res}(\xi_n) \tag{21}$$

Here I_{SDP} stands for integral (19), wherein the integration is taken over the contour C', which is dominated by the contribution of its two saddle points $\xi = \pm\pi/2$; $\mathrm{res}(\xi_n)$ are the residues of the function $f(\xi,\theta)$ in the respective poles, which are crossed by the contour C during its deformation to the contour C'.

The first region to consider is that where θ is not close to 0, π or 2π (when this condition does not hold, the poles in (19) become close to the saddle points; in this case the saddle-point method has some peculiarities (see [23,24]) and this situation is discussed below separately). The poles of the function $f(\xi,\theta)$, crossed by the contour C during its deformation to C', are

$$\xi_1 = \theta - \frac{\pi}{2} \text{ if } 0 < \theta < \pi \text{ and } \xi_2 = \frac{3\pi}{2} - \theta \text{ if } \pi < \theta < 2\pi$$

Thus, for all θ

$$-2\pi\mathrm{i}\sum_m \mathrm{res}(\xi_m) = K\,\mathrm{e}^{\mathrm{i}|\beta|(\pi-\theta)}\,\mathrm{e}^{\mathrm{i}kr\cos\theta} \tag{22}$$

The value of the integral I_{SDP} is dominated by the values of its integrand at the two saddle points

$$I_{\mathrm{SDP}} = \frac{K}{\sqrt{2\pi kr}}\left(\mathrm{e}^{\mathrm{i}(kr-(\pi/4))} f\left(-\frac{\pi}{2},\theta\right) + \mathrm{e}^{-\mathrm{i}(kr-(\pi/4))} f\left(\frac{\pi}{2},\theta\right)\right)$$

By direct calculations we obtain

$$I_{\mathrm{SDP}} = \frac{K}{\sqrt{2\pi kr}}\pi|\beta|\cot\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}kr-\mathrm{i}(\pi/4)} \tag{23}$$

The singularity in (23) is smoothed when the situation of the poles ξ_n being near the saddle points ξ_s (i.e. when θ is close to 0, π or 2π) is explicitly considered by making use of the appropriate formulae of the saddle-point method. Namely, the asymptotic expansion of I_{SDP}, valid uniformly as $\xi_n \to \xi_s$ is given by (see [22])

$$I_m(kr) \sim \mathrm{e}^{-\mathrm{i}kr\sin(\xi_s)}\left\{\pm\mathrm{i}a\pi\,e^{-krb^2}\mathrm{erfc}(\mp\mathrm{i}b\sqrt{kr}) + \sqrt{\frac{\pi}{kr}}T\right\}, \quad kr\to\infty \tag{24}$$

where the upper signs are used if $\mathrm{Im}\,b > 0$ and the lower signs are used if $\mathrm{Im}\,b < 0$, and

$$a = \lim_{\xi-\xi_n}[(\xi-\xi_n)f_m(\xi)], \quad b = \sqrt{\mathrm{i}\sin(\xi_n) - \mathrm{i}\sin(\xi_s)}, \quad T = hf(\xi_s) + \frac{a}{b}, \quad h = \sqrt{\frac{2i}{\sin(\xi_s)}}$$

Using these formulae for $\theta = \pi+\varepsilon$ (ε is small) yields $I_{\mathrm{SDP}} = 0$, and for $\theta = \varepsilon$ (ε is small) yields $I_{\mathrm{SDP}} \sim -\mathrm{i}\pi|\beta|K(1+O(\varepsilon))$. Thus, from (21) we obtain that in both cases

$$\Phi = \mathrm{e}^{\mathrm{i}kr\cos\theta}(1+O(\varepsilon)) \tag{25}$$

so that the singularity in (23) disappears. Therefore, the incident field in the region $\alpha/M < r \leqslant R$ is given by (25) when θ is close to 0, π or 2π, and by

$$I = K\,\mathrm{e}^{\mathrm{i}|\beta|(\pi-\theta)}\mathrm{e}^{\mathrm{i}kr\cos\theta} + \frac{K\pi|\beta|}{\sqrt{2\pi kr}}\cot\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}kr-\mathrm{i}(\pi/4)} \tag{26}$$

when θ is not close to 0, π or 2π. Thus, the incident sound field (16) in region I consists of the quasi-plane wave and the outgoing cylindrical wave (see Fig. 3). The appearance of the latter is due to the long range convection (or refraction) of sound by the slowly decaying mean velocity field of the vortex.

Let us note the paper [25], where by making use of the partial harmonic expansion a numerical estimate of the solution for shallow water boundary displacements by a vertical vortex is provided. Those authors are the first to endeavor to adopt the solution of [21] developed for Aharonov–Bohm effect to the sound scattering problem. In that work, an initial dislocated wave is used as the incident field. The incident field condition of this type is discussed in [1].

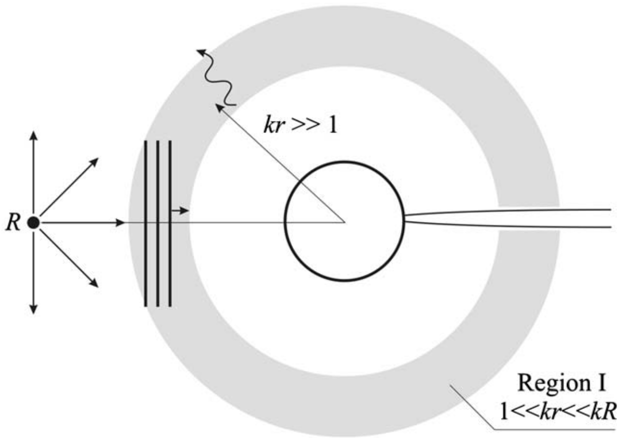

Fig. 3. The point source at the large distance R. In region I, the incident field is a sum of a quasi-plane wave, an outgoing cylindrical wave of $\cot\theta/2$ type and a smoothing wave in the narrow region behind the vortex.

3.2. Scattered field

The solution of the problem of sound scattering by the vortex is supposed to be a sum of the incident field (16) (which in region I has been evaluated to be of the form (26)) and the wave scattered by the vortex

$$\varphi = \sum_{n=-\infty}^{+\infty} (a_n J_{|\nu|}(\xi) + F_n H_{|\nu|}^{(1)}(\xi))\, \mathrm{e}^{\mathrm{i}n\theta - \mathrm{i}\omega_0 t} \tag{27}$$

where$a_n = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\pi n} H_{|\nu|}^{(1)}(kR)/4$, $\xi = kx \sim Mx$, F_n is an amplitude of the corresponding harmonic of the scattered wave. The problem thus consists in determining the amplitudes F_n for all n assuming that

$$F_n = f_n M^2 + o(M^2) \tag{28}$$

The solution of the problem will be found as follows. The solution in the subregion outside the vortex will be determined from the condition of coincidence at the boundary of the vortex with the internal solution (the pressure and normal velocity of one solution are set equal to those of another one), which is finite at the point of origin. Two main regions are distinguished (Section 2): (i) inside the vortex, $x \leq 1$ and (ii) outside the vortex, $x \geq 1$. The latter is partitioned into a near-field subregion, $x \sim 1$ and a far-field subregion $x \gg 1$, which overlap in an intermediate subregion $\xi = k\alpha x \sim 1$, where k is given by (12). The solution in the far-field subregion (a sum of the incident field and the scattering field) is to be matched with the solution in the near-field subregion. We use van Dyke's matching principle $(m,\, l)_{\mathrm{ex}}$=$(l,\, m)_{\mathrm{in}}$, i.e. the m-term inner expansion of the l-term outer expansion equals the l-term outer expansion of the m-term inner expansion [20]. These conditions in the aggregate give the complete solution of the problem for each nth harmonic. For the non-resonant case only the two-term expansion in M is required: namely $O(1)$ and $O(M^2)$ terms are retained. The solution is a sum of the $O(M^2)$ terms for all harmonics. The resonant case requires more terms in the expansion but for only one harmonic, which is resonant.

Note, that for the matching procedure it is very important that we know the two-term solution for each nth harmonic in the far-field subregion (the solution of the form (27) of Eq. (12)) since it enables us to obtain terms of the form $(m,1)_{\mathrm{ex}}$ and $(m,2)_{\mathrm{ex}}$ in van Dyke's notations as $\xi \to 0$.

3.2.1. Outer region, near-field subregion

Let us determine the solution of Eq. (11) in the region near the vortex ($x \sim 1$): since Eq. (11) is to be valid for $O(1)$ and $O(M^2)$ terms it means we obtain the two-term solution $(m,1)_{\mathrm{in}}$ and $(m,2)_{\mathrm{in}}$. Let us determine the two leading terms of the expansion in the Mach number.

In the following analysis, the subscript n in the notation for nth harmonic is dropped. Using the method of successive approximations, one finds that the nth harmonic amplitude for the potential φ is as follows:

$$\varphi = A_1\left[1 - \frac{M^2\omega^2 x^2}{4}\right] + A_2\left[\ln x + \frac{M^2\omega^2 x^2}{4}(1-\ln x) - \frac{M^2}{4x^2}\right], \quad n = 0.$$

$$\varphi = A_1 x^{|\nu_0|}\left[1 - M^2\frac{A_2}{A_1}\frac{\omega^2}{2}\ln x\right] + A_2 x^{-|\nu_0|}\left[1 + M^2\frac{A_1}{A_2}\ln x\right] - A_1\frac{M^2\omega^2}{8}, \quad |n| = 1.$$

$$\varphi = A_1 x^{|\nu_0|} + A_2 x^{-|\nu_0|} + \frac{M^2}{4}\left[-\frac{A_1\omega^2}{|\nu_0|+1}x^{|\nu_0|+2} + \frac{A_1(n^2+|\nu_0|)}{|\nu_0|-1}x^{|\nu_0|-2} + \frac{A_2\omega^2}{|\nu_0|-1}x^{-|\nu_0|+2} - \frac{A_2(n^2-|\nu_0|)}{|\nu_0|+1}x^{-|\nu_0|-2}\right], \quad |n| \geq 2. \tag{29}$$

where A_1, A_2 are independent constants, A_1 corresponds to the increasing part of the solution and A_2 corresponds to the decreasing part of the solution. Knowing the amplitude φ near the vortex, we can find straightforwardly the amplitudes of the nth harmonic for the radial velocity u_r and the pressure p.

3.2.2. Region inside vortex

In the region inside the vortex, the propagation of sound perturbations is governed by the system of equations (6). To find the pressure amplitudes in two-term expansion in M one has to solve Eq. (8). The solution finite at the point of origin can be easily found using the method of successive approximations and is given to two leading terms of the expansion in the Mach number by

$$p = C_1 x^{|n|}\left(1+\frac{M^2}{4}x^2\frac{|n|-\Psi}{|n|+1}\right) \tag{30}$$

where C_1 is an unknown constant of the inner solution (the second constant is zero because of the condition of finitude at the point of origin).

3.2.3. Matching of solutions at the vortex boundary

In order to obtain a relation between the constants A_1, A_2 and C_1, the solution for pressure and normal velocity in the region inside the vortex and the corresponding solution in the near-field subregion of the outer region are to coincide at the vortex boundary $x=1$. Note, that van Dyke's matching principle is not used here, it is necessary to match the solutions only in the overlapping region in the far field. Determining an expression for the radial velocity from (29) and (30) with the help of (6) and joining the obtained solutions for the pressure and the radial velocity at the vortex boundary, for each case $|n| \geq 2$, $|n|=1$ and $n=0$ one gets a system of two equations for three unknown constants A_1, A_2 and C_1. Eliminating C_1 in each system, one determines equations, relating A_1 and A_2 to each other:

$$\begin{cases} A_2 = \frac{3}{4}M^4\omega^2 A_1[1+O(M^2)], & n=0 \\ A_1 = \omega\,\mathrm{sgn}(n)A_2, & |n|=1 \\ A_1 = A_2[1-(n-\omega)\mathrm{sgn}(n)+O(M^2)], & |n|\geq 2 \end{cases} \tag{31}$$

Thus, there is only one unknown constant, A_1, in the near subregion and only one unknown constant F_n in the far subregion. It is easy to see that for $|n| \geq 2$ the constants A_1 and A_2 are of the same order, if the frequency is not close to an eigenfrequency of the vortex (i.e. if the non-resonant scattering is considered). The case of resonance is considered in Section 4 using the results of Appendix A.

3.2.4. Matching with the far-field solution

Using the method of matched asymptotic expansions and van Dyke's matching principle, one obtains the following:

For $n=0$:

In this case the two-term outer subregion solution (27) is $\varphi = a_0 J_0(\xi) + f_0 M^2 H_0^{(1)}(\xi)$ and the two-term inner subregion solution is determined by (29) and (31). Using the asymptotic forms for the Bessel functions and van Dyke's matching principle $(1,1)_{\mathrm{ex}}=(1,1)_{\mathrm{in}}$ we obtain $A_1 = a_0$, $A_2 = 0$. Using $(1,2)_{\mathrm{ex}}=(2,1)_{\mathrm{in}}$ yields $f_0 = 0$.

For $|n|=1$:

Suppose that the constants are $O(M)$: $A_1 = k\alpha\overline{A}_1$, $A_2 = k\alpha\overline{A}_2$. Van Dyke's matching principle $(1,1)_{\mathrm{in}}=(1,1)_{\mathrm{ex}}$ then yields $\overline{A}_1 = a_n/2$; $\overline{A}_2$ is given by (31), so that $\overline{A}_2 = a_n\mathrm{sgn}(n)/2\omega$. To determine f_n we use the equation $(2,1)_{\mathrm{in}}=(1,2)_{\mathrm{ex}}$, which is as follows:

$$\overline{A}_1 k\alpha x + \frac{\overline{A}_2 k\alpha}{x} = a_n\left(\frac{k\alpha x}{2}\right) - \frac{2iM^2}{\pi k\alpha x}f_n$$

Thus, $f_n = \mathrm{i}\pi\omega a_n \mathrm{sgn}(n)/4$, where a_n is defined after (27). For $n=\pm 1$ the amplitudes are $F_n = -\pi\omega M^2\mathrm{sgn}(n)H_{|\nu|}^{(1)}(kR)/16$ and the scattered wave in the leading approximation is given by

$$\varphi = (f_1\,\mathrm{e}^{\mathrm{i}\theta} + f_{-1}\,\mathrm{e}^{-\mathrm{i}\theta})M^2 H_1^{(1)}(\xi)\,\mathrm{e}^{\mathrm{i}\omega t} = -\frac{\mathrm{i}\pi}{8}\omega M^2 H_1^{(1)}(kR)H_1^{(1)}(kr)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\sin\theta \tag{32}$$

For $|n| \geq 2$:

Suppose that $A_2 = (k\alpha)^{|n|}\overline{A}_2$, $A_1 = (k\alpha)^{|n|}\overline{A}_1$ (according to (31) they are of the same order). Evaluating the series of $\varphi = a_n J_{|\nu|}(\xi) + f_n M^2 H_{|\nu|}^{(1)}(\xi)$ as $\xi \to 0$ and using (29) we perform the matching according to van Dyke's matching principle. From the equation $(1,1)_{\mathrm{ex}}=(1,1)_{\mathrm{in}}$ we obtain $\overline{A}_1 = a_n/(|n|-1)!2^n$, $\overline{A}_2 = 0$. Using $(1,n)_{\mathrm{ex}}=(n,1)_{\mathrm{in}}$ yields $f_n = 0$. It means that $F_n = o(M^2)$ for $|n| \geq 2$.

The results of the matching procedure can now be summarized. The aim is to discover the leading-order term in an expansion of the scattered field in powers of Mach number. We therefore retain in the second term of (27) the harmonics with $n=\pm 1$ only, which are of order M^2, and neglect the other harmonics (since the scattering amplitude coefficients F_n are negligibly small for $n=0,\ n\geq 2$). This means that the scattering field is determined by expression (32). The incident field corresponding to the first term in (27) is represented by the complete series. Note that the scattered wave, given by

(32), is a standard dipole field and does not contain any singularities. It represents the re-emission of sound by the vortex, when it is forced to oscillate by the incident wave field (as noted in [5,10] for the plane wave formulation).

The complete solution in the far field is the sum of the scattered field (32) and the incident field due to the point source (16):

$$\varphi = \sum_{n=-\infty}^{+\infty} \frac{e^{-i\pi n}}{4i} H^{(1)}_{|\nu|}(kR) J_{|\nu|}(kr)\, e^{in\theta - i\omega_0 t} - \frac{i\pi}{8}\omega M^2 H^{(1)}_1(kR) H^{(1)}_1(kr)\, e^{-i\omega_0 t} \sin\theta \tag{33}$$

Further insight can be gained by examining this expression in the region $\alpha/M < r \ll R$, where the series in the first term of (33) turns out to be determined explicitly. By using Eq. (26) for the incident field when θ is not close to 0, π, or 2π, one gets

$$\varphi = K\, e^{i\beta(\pi-\theta)}\, e^{ikr\cos\theta} - \frac{K}{\sqrt{2\pi kr}}\pi\beta\left(\sin\theta - \cot\left(\frac{\theta}{2}\right)\right) e^{ikr - i(\pi/4)}, \quad K = \frac{e^{-i\omega_0 t}}{4i}\sqrt{\frac{2}{\pi kR}}\, e^{i(kR-(\pi/4))} \tag{34}$$

where the amplitude K relates to the location of the point source and the point of observation. Naturally, in a narrow parabolic region behind the vortex the field has a more complicated structure, expressed through the Fresnel integrals (24), which smooth the singularity corresponding to ctg($\theta/2$). This result helps to understand where the previous solutions break down. It turns out that the majority of papers using polar coordinates correctly describe the sound–vortex interaction field by representing the field as a plane wave and the two terms of (1). However, the solution is close to the form of (1) in region I only. The singularity corresponding to cot($\theta/2$) is an attribute of the incident field, or more precisely, the incident field transformation in the slowly decaying mean velocity field of the vortex. The wave front deformation in the far field, and in a narrow parabolic region behind the vortex, appears to be correctly obtained by using Cartesian coordinates. However, all comparisons can be made only in region I, whereas outside it this conclusion is invalid. When one approaches the vortex, or conversely moves away from region I, the complete expression (33) should be used.

Thus, there are three analytical solutions to the governing equations in the far field, which do not use doubtful mathematical procedures: (i) the solution obtained by Berry et al. [21], (ii) the solution with a point source, and (iii) the solution with a plane wave at a finite distance around the vortex [1]. They coincide in the leading order in M but differ in the order of M^2. The first solution is obtained for the whole far-field region and is a sum of a quasi-plane wave incident from the left and an outgoing cylindrical wave. The second solution is obtained in an explicit form in region I (Fig. 3) and looks like the solution of Berry et al. In the far field beyond the region I this solution is expressed in terms of an infinite series. The third solution is also obtained in an explicit form only in Region I and is a sum of quasi-plane wave incident from the left and outgoing and incident cylindrical waves of the order of M^2.

The question arises as to what can be expected to obtain in an experiment (physical or numerical). The answer to this question appreciably depends upon the formulation of the experiment and in particular upon the width of the region, where the wave is defined. If the experimental region is narrow and the left boundary, on which the condition of incident plane wave is stated, goes entirely in the Region I, then the scattered fields will be close to predicted in (i) and (ii). If this experimental region is wide then the solution with an incident plane wave and the solution with a point source placed farther to the left will be different, because the solution with a point source may differ from the (quasi-)plane wave stated on the left boundary of the experimental region.

We do not discuss the condition of replacement of the small $\sim 1/r$ vortex-induced field with no flow boundary conditions on the boundaries of the experimental region because it renders the formulation of the experiment significantly more difficult. However, if the plane wave boundary condition is stated where it is known to be correct, i.e. in the fluid at rest, it leads to the third formulation of the experiment, where the vortex is encircled with a weak vortex sheet with the same total circulation but of the opposite sign. This results in whole region outside of the vortex sheet absence of flow. The formulation with a plane wave posed in this region seems to lead to solution (iii).

It should be borne in mind that all these differences in the scattered field are of order of M^2. The singularity in the forward scattered direction appears in all three cases, so that if the accuracy of the experiment allows only the existence of a dislocated wave to be determined, the results of the first and second experiments seem to be indistinguishable. In the third experiment there may appear the new singularity which may be experimentally determined. This analysis does not settle the question but rather pave the way for the further discussion and research.

4. Resonant scattering

The Rankine vortex is known to be an oscillatory system that can emit sound in a compressible fluid (see [5,9,17]). Sound scattering may cause resonant excitation of oscillations, and the scattered field (re-emission by excited degrees of freedom) may increase by many times. In this case, eigenfrequencies of the system become poles of the scattering amplitude. The eigenfrequencies of the emitting vortex have imaginary parts that correspond to instabilities—see [16,9,10]. Since there is no flow along the vortex axis, such an instability cannot be convective [18]. At the same time, the instability is so weak that the nonstationary problem formulation with the model of two-dimensional Rankine vortex is quite possible because during the time reciprocal to the instability increment the effects of viscosity may change entirely the general character of fluctuations.

Besides, the spectral problem formulation (with a real-valued frequency of excitation) is quite possible as an exact mathematical formulation. As shown in [10], the solution is finite even in the resonant case. This constitutes the obvious difference between an exact mathematical formulation and a numerical experiment where initial disturbances for the unstable harmonics are always present so that one does not succeed in obtaining the spectral solution for the real-valued frequency because the solution oscillates with unrestrictedly large amplitudes due to the instability [19]. Despite very special role of spectral problem for unstable system, this solution can be used to describe the first stage of the evolution of the undisturbed vortex under periodical excitation (initial-value problem). The spectral solution and eigensolution with the complex-valued frequency (taken in the combination with zero amplitude of vortex boundary in the initial moment) can be used to describe the evolution of the undisturbed vortex until the unstable eigensolution destroys the flow as a whole [10].

Since the eigenfrequencies of the emitting vortex have imaginary parts that correspond to instabilities—see [16,9,10]—the denominator of the scattering amplitude is not zero at the real-valued incident frequency, and the amplitude is given by

$$F_n \sim \frac{\mathrm{i}\delta_n}{\omega-\omega_n-\mathrm{i}\delta_n} \tag{35}$$

Here ω is the dimensionless incident frequency, and ω_n and δ_n are the real and imaginary parts of the eigenfrequency. Expression (35) is obtained in Kopiev and Leontiev [10], where $\delta_n \sim M^{2n}$ is an amplification coefficient for each unstable harmonic, calculated for the first time for $n=2$ in [16]. This structure of the solution demonstrates that the steady-state amplitude is finite; the situation is analogous to scattering by quasi-discrete levels of energy in quantum mechanics [26]. Furthermore, resonant scattering is possible only for $n \geq 2$ (eigenoscillations of the vortex exist for these n). Non-resonant scattering for these harmonics is small—it is $O(M^4)$—and can be neglected. As follows from Eq. (35), when the incident-wave frequency coincides with ω_n, the scattering amplitude is of order unity, which is larger than its non-resonant contribution (32).

In [15] this conclusion was analyzed in detail to $O(M^2)$; it was shown that when the incident frequency ω coincides with the eigenfrequency of the incompressible Rankine vortex, $\omega_n = n-1$, the scattering amplitude is smaller than unity, and in the case of $n=2$ it increases only to the value $\sim M^2$. Clearly, there is a contradiction with the previous paragraph. The objective of the discussion that follows is to resolve this contradiction.

We proceed by expanding all variables to $O(M^4)$. This approximation is sufficient only for the harmonic $n=2$ (for harmonics $n \geq 3$ more terms are necessary); therefore, it is $n=2$ that is considered below. For this n the exact equation (7) inside the vortex, and the exact equation (10) in the intermediate region outside the vortex ($x \leq 1/M$) will be considered; the solution of the latter equation must be asymptotically matched with the solution of (12) ($x \geq 1/M$). It appears that for the matching we could restrict ourselves to the leading approximation in (12).

Since the exact eigenfrequency of the compressible vortex $\omega_i = \omega_n + \mathrm{i}\delta_n$ must be used in (35), let us find it for n=2 (for higher n we would need to determine the higher order terms). The outgoing-wave solution to Eq. (12) should be chosen (as in [16]); this corresponds to setting a_n=0 in Eq. (27). Consider the far-field subregion of the region outside the vortex. We restrict ourselves to the leading approximation in Eq. (21) with the solution of the form $\varphi_{rad} \sim H^{(1)}_{|n|}(\xi)$, because the leading-order solution $(m,1)_{\mathrm{ex}}$ appears to be sufficient for matching. Expanding $H^{(1)}_{|n|}(\xi_i)$, $\xi_i = k_i \alpha x$, $k_i^2 = M^2\omega_i^2/\alpha^2$ in a series for small x up to accuracy of $O(M^4)$ we obtain $(3,1)_{\mathrm{ex}}$. In order to obtain $(1,3)_{\mathrm{in}}$ we use Eq. (A.1) from Appendix A. Then van Dyke's matching principle $(3,1)_{\mathrm{ex}}=(1,3)_{\mathrm{in}}$ yields a relation between the amplitudes A_1 and A_2 in Eq. (A.1)

$$\frac{A_1}{A_2} = \frac{\mathrm{i}\omega^4 M^4}{16}\left(\frac{\pi-\mathrm{i}}{2} + \mathrm{i}C_\gamma - \mathrm{i}\ln\left(\frac{2}{\omega M}\right)\right) \tag{36}$$

where $C_\gamma = 0.5772\ldots$ is the Euler constant.

Let us determine the same relation between A_1 and A_2, using the matching of the outer solution of Eqs. (7) and (10), calculated to $O(M^4)$. We take Eq. (A.1) in the outer region and Eq. (A.2) in the inner region. Matching the pressures and normal velocities at the vortex boundary yields (A.3) as follows:

$$\frac{A_1}{A_2} = (\omega_i-1) + \frac{M^2}{12}(\omega_i^3+8\omega_i^2-28\omega_i+20) + \frac{M^4}{1152}(2\omega_i^5+\omega_i^4(6\gamma+45)-\omega_i^3(280+288\gamma)+\omega_i^2(588\gamma-860)+\omega_i(408\gamma+2864)-1740-708\gamma) \tag{37}$$

After setting (37) equal to (36), one obtains the eigenfrequency for n=2,

$$\omega_2 = 1-\frac{M^2}{12} - M^4\left(\frac{67}{1152}+\frac{C_\gamma}{16}-\frac{\mathrm{i}\pi}{32}+\frac{\gamma}{192}+\frac{1}{16}\ln\frac{M}{2}\right) \tag{38}$$

This expression coincides with that of [16]; it should be noted, however, that there seems to be a misprint in [16]: the coefficient in the first term in the brackets must be 67/1152, not 67/1162.

Let us consider scattering by the vortex when the incident-wave frequency is close to (38):

$$\omega = 1-\frac{M^2}{12} - wM^4, \tag{39}$$

where $w = O(1)$ is an arbitrary constant, i.e. the frequency of the incident wave differs from the real part of the eigenfrequency for n=2 by an amount of $O(M^4)$. The relation between the coefficients A_1 and A_2 in (37) remains valid. Let us determine the same relation between A_1 and A_2, by matching the solutions in the far field. As above we restrict ourselves to the leading approximation in Eq. (12) because the leading-order solution $(m,1)_{ex}$ appears to be sufficient for the matching. However, in this case we should take the incident field into account. Expanding $\varphi_2 = a_2 J_{|n|}(\xi)+F_2 H^{(1)}_{|n|}(\xi)$, $n=\pm 2$ in a series for small x up to $O(M^4)$ we obtain $(3,1)_{ex}$. In order to obtain $(1,3)_{in}$ we use Eq. (A.1) from the Appendix A. Then van Dyke's matching principle $(1,3)_{in}=(3,1)_{ex}$ yields a relation between the amplitudes A_1 and A_2 in Eq. (A.1). Therefore, the relation between A_1 and A_2 changes and instead of (36) one gets

$$\frac{A_1}{A_2} = \frac{M^4}{32}\left(\mathrm{i}\pi+1-2C_\gamma+2\ln\left(\frac{2}{M}\right)+\mathrm{i}\pi\frac{a_2}{F_2}\right) \tag{40}$$

After setting (40) equal to (37) for the scattering amplitude F_2 we finally obtain

$$F_2 = a_2\frac{\frac{\mathrm{i}\pi}{32}M^4}{M^4\left(\frac{67}{1152}+\frac{C_\gamma}{16}+\frac{\gamma}{192}+\frac{1}{16}\ln\left(\frac{M}{2}\right)\right)-M^4 w-\frac{\mathrm{i}\pi}{32}M^4}. \tag{41}$$

We can see that the resonant scattering amplitude in F_2 is indeed of $O(1)$ rather than $O(M^2)$, and has the precise structure described in (35) (as was predicted in [10]). However, the resonance evidently occurs at a finite value of w in expression (39), i.e. at a frequency different from the resonant one by terms of $O(M^4)$. If the incident frequency coincides with an eigenfrequency of the incompressible vortex (coincidence with (38) in the leading-order terms), the largest term in the denominator of (35) is $\omega-\omega_2 \sim M^2$, and the numerator $\delta_2 \sim M^4$ is not canceled. Therefore, the true resonance is not achieved at the incompressible eigenfrequency, and the corresponding scattered amplitude is evidently $O(M^2)$, as predicted in [15]. Note that resonant scattering corresponds to a rotating field $\sim \exp(-\mathrm{i}\omega t+\mathrm{i}n\theta)$ but a standing wave along r. This accords with the condition of no energy flux to infinity because we consider steady-state oscillations with real frequency.

It should be stressed that van Dyke's matching procedure, used in the resonant scattering problem, requires only the terms $(1, n+1)_{ex}$ for matching. Consequently, in Eq. (12) we can set $\nu=n$ without affecting the results of resonant scattering. Thus, unlike the case of non-resonant scattering, any incident sound field that coincides in the leading-term approximation with that of a point source at large but finite distance (e.g. a plane sound wave) allows the resonant scattering amplitude to be correctly determined.

5. Conclusion

The well-known problem of long-wavelength sound scattering by a Rankine vortex has been reviewed with the aid of small-M asymptotic methods. The problem is basic for investigating sound–vortex interactions, but there has been some confusion so far. Causes of this confusion were examined in [1]. It is shown here that instead of the d'Alembert equation with right-hand part (sources), the convective equation with nonuniform coefficients should be considered; and instead of the plane incident wave condition (a plane wave is not the solution of the governing equations at any distance), the condition of a point source far from the vortex should be posed.

To aid in classifying the existing solutions for non-resonant scattering, a point source placed at large but finite distance from the vortex has been considered in this paper. This allows one to pose a correct mathematical formulation of the problem and to compare different approaches and results (see Section 3). The scattered field in the region far outside the vortex is determined as the solution of the convective wave equation, and van Dyke's matching principle is used to match with the vortex region solution.

Resonant scattering is considered under the new formulation (with the point source) and an exact solution is obtained, which unifies previous results and resolves the existing contradiction between them. It is found that the leading approximation to the incident field (i.e. a plane wave) is sufficient for matching in the resonant case, but a three-term $O(M^4)$ solution in the inner region is necessary.

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Appendix A

The solution of Eq. (10) in the third approximation (i.e. when terms of $O(M^4)$ are retained) is straightforwardly determined by using the method of successive approximation and is as follows:

$$\begin{aligned}\varphi_{in} = {} & A_1x^2+\frac{A_2}{x^2}+M^2\left[A_1\left(\frac{3}{2}-\frac{\omega x^2}{4}-\frac{\omega^2x^4}{12}+\omega x^2\ln(x)\right)+A_2\left(\frac{\omega^2}{4}-\frac{1}{6x^4}-\frac{\omega}{4x^2}-\frac{\omega\ln x}{x^2}\right)\right]\\ & +\frac{M^4}{1152x^6}\left\{A_1\left[x^4\left(432+12(\gamma-1)\left(18+4x^6\omega^2+3x^4\omega(\omega+4)-24x^2(2\omega+3)\right)\right.\right.\right.\\ & \left.-x^2\omega\left(1872-x^2\omega\left(204+x^2\omega(56+3x^2\omega)\right)\right)\right.\end{aligned}$$

$$+48\ln x\Big(18+3\gamma\Big(6-x^4\omega(\omega+4)\Big)+2x^2\omega\Big(18+x^2\Big(6-\omega(7+x^2\omega)\Big)\Big)+12x^4\omega^2\ln x\Big)\Big)\Bigg]$$
$$+A_2[36(1-\gamma)+2x^2\Big(6(\gamma-1)\Big(8+\omega\Big(16+3x^2(4+\omega-4x^2\omega)\Big)\Big)$$
$$+\omega\Big(104+3x^2\omega\Big(34-3x^2\omega(20-x^2\omega)\Big)\Big)\Big)$$
$$+24x^2\ln x\Big(8+x^2\Big(6\gamma(\omega+4)-24+\omega\Big(28-3x^2\omega(4+x^2\omega)\Big)\Big)+24x^2\omega\ln x\Big)\Bigg]\Bigg\} \tag{A.1}$$

In the same way the solution of Eq. (7) is obtained. It is given by

$$p=C_1x^2\left[1+\frac{M^2}{12}x^2(2-\Psi)+\frac{M^4}{384}x^2(16(\gamma-1)(2-\Psi)+x^2(16+(\Psi-4)(6\gamma+\Psi)))\right] \tag{A.2}$$

From these equations with the help of (6) and (9) the pressure and radial velocity in both regions are determined. The pressure and radial velocity are to coincide at the vortex boundary; this requirement implies a system of two equations for three unknown constants A_1, A_2 and C_1. Eliminating C_1 in each system, we obtain the relation between A_1 and A_2:

$$\frac{A_1}{A_2}=(\omega-1)+\frac{M^2}{12}(\omega^3+8\omega^2-28\omega+20)+\frac{M^4}{1152}(2\omega^5+\omega^4(6\gamma+45)-\omega^3(280+288\gamma)+\omega^2(588\gamma-860)+\omega(408\gamma+2864)-1740-708\gamma) \tag{A.3}$$

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